

**C5 APPLICATIONS OF INTEGRATION**

**Integration** may be used to solve problems such as finding the area under a curve or determining a variable quantity when its rate of change is known.

Topic C5 illustrates how the technique of integration is applied to solving the problem of finding the **area under a curve** for polynomial functions.

Topic C5 assumes you have studied Topic C4 The Rules of Integration (C4.7 The Definite Integral, is particularly relevant).

**LESSON PLAN**

<b>Lesson No.</b>	<b>LESSON TITLE</b>	<b>Page</b>	<b>Examples</b>	<b>SAQs</b>
1	<b>C5.1 Area Under a Curve 1</b>	2	5.1 - 5.2	C5.1
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**C5.1 THE AREA UNDER A CURVE**

For some applications we wish to determine the area under a curve. For example, the area between given limits under:

- velocity/time curves give distance travelled
- force/distance curves give work done
- velocity/current curves give power dissipated

Provided there is a known relationship between the variables, then the area under the curve may be calculated using integral calculus.

When the relationship between the variables is not known or the function relating the variables cannot be integrated exactly, a numerical method, such as Simpson's rule, can be used to give an approximate area. Numerical methods of integration are not included in these notes.

**Note:** The diagrams and mathematical explanations in the following sections:

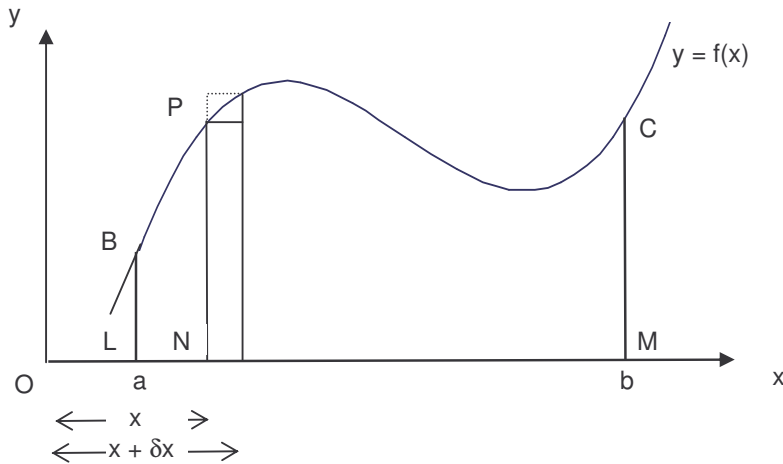
**C5.1.1** The Integral as the Area Under a Curve

**C5.1.2** The Integral as the Limit of a Sum

are given as **background information** to indicate why integration can be used to find the area under a curve.

**C5.1.1 THE INTEGRAL AS THE AREA UNDER A CURVE**

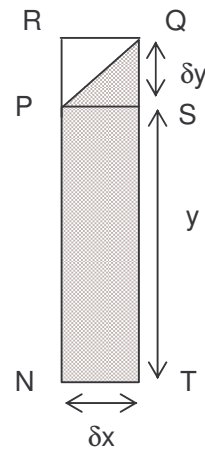
The diagram below shows an area BCML, bounded by the curve  $y = f(x)$ , the x-axis and the ordinates BL at  $x = a$  and CM at  $x = b$ .



Let  $P(x, y)$  be any point on the curve  $y = f(x)$  where  $a \leq x \leq b$ . If  $A$  is the area BPNL, then  $A$  is a function of  $x$ . A small change  $\delta x$  in the value of  $x$ , produces a corresponding small change  $\delta y$  in the value of  $y$  and also a corresponding change  $\delta A$  in the value of  $A$ .

The diagram on the right shows that the shaded area  $\delta A$  satisfies:

$$\begin{aligned} \text{Area PSTN} < \delta A < \text{Area RQTN} \\ \Rightarrow y \delta x < \delta A < (y + \delta y) \delta x \\ \Rightarrow y < \frac{\delta A}{\delta x} < y + \delta y \end{aligned}$$



In the limit, as  $\delta x \rightarrow 0$  it follows that  $\delta y \rightarrow 0$  and so

$$y \leq \frac{dA}{dx} \leq y \Rightarrow y = \frac{dA}{dx}$$

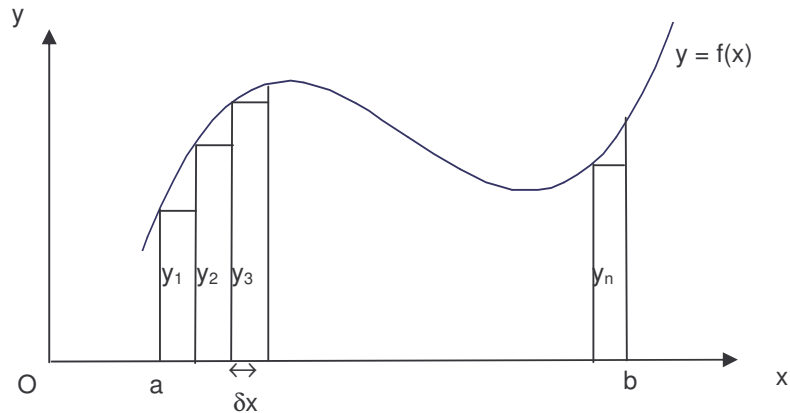
If we integrate  $y = \frac{dA}{dx}$  with respect to  $x$

$$\text{then } A = \int y \, dx \quad (\text{indefinite integral})$$

$$\begin{aligned} \text{Hence } \int_a^b f(x) \, dx &= \int_a^b y \, dx = [A]_a^b \\ &= (\text{value of } A \text{ when } x = b) - (\text{value of } A \text{ when } x = a) \\ &= \text{area of BCML} - 0 \\ &= \text{area of BCML} \end{aligned}$$

Hence, the **area under the curve**  $y = f(x)$  between  $x = a$  and  $x = b$  is the **definite integral**  $\int_a^b y \, dx$ .

**C5.1.2 THE INTEGRAL AS THE LIMIT OF A SUM**



The diagram above shows the area under the curve  $y = f(x)$  between  $x = a$  and  $x = b$ . The area under the curve is divided into  $n$  rectangles, each of width  $\delta x$  and height the value of  $y$ , ie.  $y_1, y_2, \dots, y_n$ . Thus the **area** of a typical rectangle is  $y\delta x$ .

If  $\delta x$  is very small, the area under the curve can be approximated by the sum of the areas of the  $n$  rectangles. Hence the total area under the curve is approximately

$$A = y_1\delta x + y_2\delta x + y_3\delta x + \dots + y_n\delta x = \sum_{i=1}^n y_i\delta x$$

which can be written as  $A = \sum_{x=a}^b y \delta x$

As the number of rectangles  $n$  increases, the width  $\delta x$  decreases and the area  $A$  becomes a closer approximation to the exact area under the curve. Thus the exact area is given by

$$A = \lim_{\delta x \rightarrow 0} \sum_{x=a}^b y \delta x$$

Since the area under the curve  $y = f(x)$  is given by the definite integral  $\int_a^b y \, dx$  we can conclude that

$$\int_a^b y \, dx = \lim_{\delta x \rightarrow 0} \sum_{x=a}^b y \delta x$$

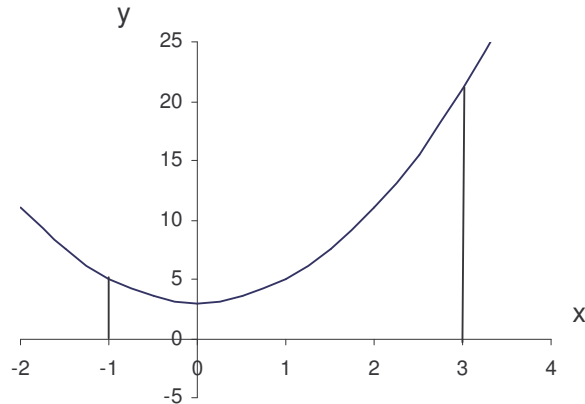
The following two examples show how integration is used to find the area, within given limits, under the curve described by simple quadratic functions.

**EXAMPLE 5.1**

Find the area bounded by the curve  $y = 2x^2 + 3$  and the x-axis between  $x = -1$  and  $x = 3$ .

**Solution:**

Graph of  $y = 2x^2 + 3$



$$\begin{aligned}
 \text{Area} &= \int_{-1}^3 (2x^2 + 3) dx \\
 &= \left[ \frac{2x^3}{3} + 3x \right]_{-1}^3 \\
 &= \left( \frac{2(3)^3}{3} + 3(3) \right) - \left( \frac{2(-1)^3}{3} + 3(-1) \right) \\
 &= (18 + 9) - \left( -\frac{2}{3} - 3 \right) \\
 &= 27 + 3\frac{2}{3} \\
 &= 30\frac{2}{3} \text{ square units}
 \end{aligned}$$

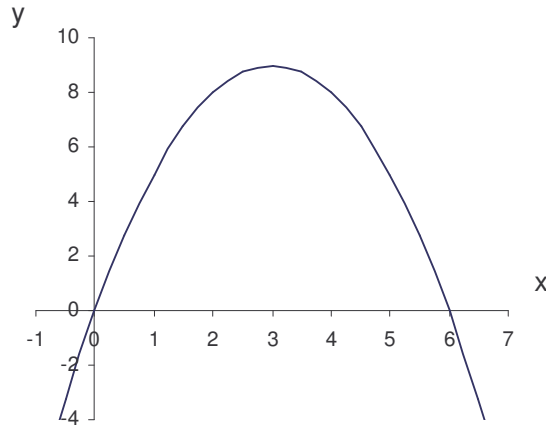
**EXAMPLE 5.2**

Find the area between the curve  $y = -x^2 + 6x$  and the x-axis.

**Solution:**

The curve cuts the x-axis where  $-x^2 + 6x = 0$   
 $-x(x - 6) = 0$   
 $\Rightarrow x = 0, 6$

Graph of  $y = -x^2 + 6x$



$$\begin{aligned}
 \text{Area} &= \int_0^6 (-x^2 + 6x) dx \\
 &= \left[ -\frac{x^3}{3} + 3x^2 \right]_0^6 \\
 &= \left( -\frac{6^3}{3} + 3(6)^2 \right) - (0) \\
 &= -72 + 108 \\
 &= 36 \text{ square units}
 \end{aligned}$$

**SAQ C5.1**

- (a) Find the area bounded by the curve  $y = 3x^2$  and the x-axis between  $x = 1$  and  $x = 3$ .
- (b) Find the area between the curve  $y = -x^2 + 6x - 5$  and the x-axis.

## C5.2 THE AREA UNDER A CURVE 2

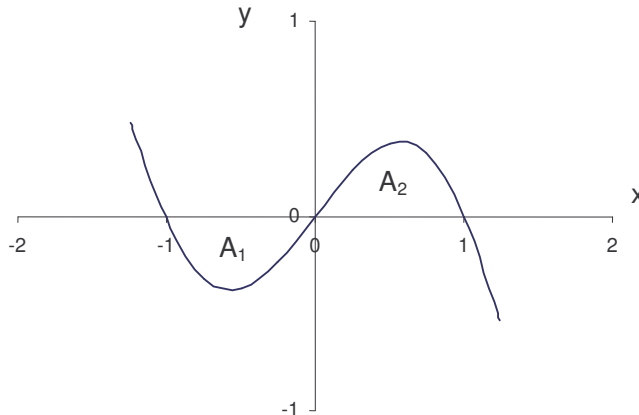
**EXAMPLE 5.3**

Find the area enclosed between the curve  $y = x(1 - x)(1 + x)$  and the x-axis.

**Solution:**

The curve cuts the x-axis where  $x(1 - x)(1 + x) = 0 \Rightarrow x = 0, 1, -1$

Graph of  $y = x(1 - x)(1 + x)$



In this example the area  $A_1$  lies **below the x-axis**. Between  $x = -1$  and  $x = 0$ , the values of  $y = x(1 - x)(1 + x)$  are negative, hence the definite integral is **negative** (as shown in Calculation 1 below). In this case, the area  $A_1$  is the absolute value of the integral.

By determining the points at which the curve **cuts the x-axis** and sketching the graph, the areas **above** and **below the x-axis** can be clearly identified. To obtain the correct value for the **total area enclosed**, we must evaluate the areas **above** and **below** the x-axis **separately**, then take the sum of the individual areas.

First, simplify the function:  $y = x(1 - x)(1 + x) = x(1 - x^2) = x - x^3$

**Calculation 1: ( $-1 < x < 0$ )**

$$\begin{aligned} & \int_{-1}^0 (x - x^3) dx \\ &= \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_{-1}^0 \\ &= (0) - \left( \frac{(-1)^2}{2} - \frac{(-1)^4}{4} \right) \\ &= -\frac{1}{2} + \frac{1}{4} \\ &= -\frac{1}{4} \end{aligned}$$

Thus,  $A_1 = \frac{1}{4} = 0.25$  square units and  $A_2 = \frac{1}{4} = 0.25$  square units

**Calculation 2: ( $0 < x < 1$ )**

$$\begin{aligned} & \int_0^1 (x - x^3) dx \\ &= \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 \\ &= \left( \frac{(1)^2}{2} - \frac{(1)^4}{4} \right) - (0) \\ &= \frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

Hence, **total enclosed area** =  $A_1 + A_2 = 0.25 + 0.25 = 0.5$  square units

**Important Note:** An attempt to calculate the total area as a **single integral**

$\int_{-1}^1 (x - x^3) dx$  would give the answer **0**, which is clearly **incorrect**.

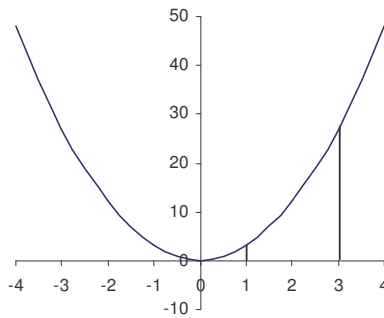
**SAQ C5.2**

- (a) Find the total area enclosed by the curve  $y = x(x - 1)(x - 3)$  and the x-axis.
- (b) Find the total area enclosed by the curve  $y = (x - 1)(x - 2)$  and the x-axis between  $x = 0$  and  $x = 3$ .

**C5.3 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS**

**SAQ C5.1**

(a) Graph of  $y = 3x^2$

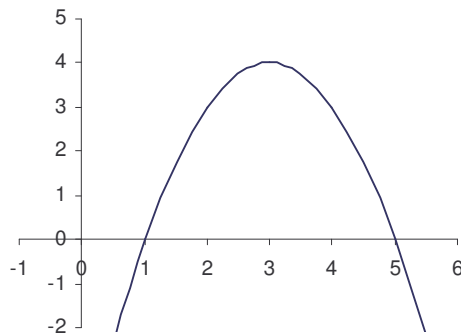


$$\begin{aligned} \int_1^3 3x^2 dx &= [x^3]_1^3 \\ &= 3^3 - 1^3 \\ &= 26 \end{aligned}$$

**Area = 26 square units**

(b) Curve cuts the x-axis where:  $-x^2 + 6x - 5 = 0$   
 $x^2 - 6x + 5 = 0$   
 $(x - 1)(x - 5) = 0$

$$\Rightarrow x = 1, x = 5$$

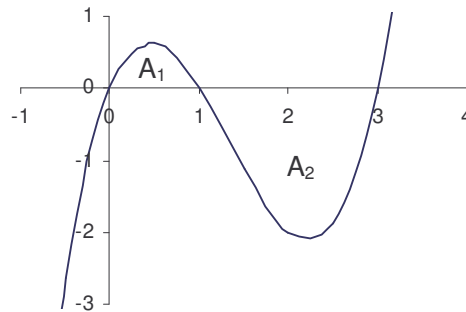


$$\begin{aligned} \int_1^5 (-x^2 + 6x - 5) dx &= \left[ -\frac{x^3}{3} + 3x^2 - 5x \right]_1^5 \\ &= \left( -\frac{5^3}{3} + 3(5)^2 - 5(5) \right) - \left( -\frac{(1)^3}{3} + 3(1)^2 - 5(1) \right) \\ &= \left( -\frac{125}{3} + 75 - 25 \right) - \left( -\frac{1}{3} + 3 - 5 \right) \\ &= 8\frac{1}{3} + 2\frac{1}{3} \\ &= 10\frac{2}{3} \end{aligned}$$

**Area =  $10\frac{2}{3}$  square units**

**SAQ C5.2**

(a) Curve cuts the x-axis where:  $x(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3$



$$y = x(x-1)(x-3) = x^3 - 4x^2 + 3x$$

$$\begin{aligned} \int_0^1 (x^3 - 4x^2 + 3x) dx &= \left[ \frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right]_0^1 \\ &= \left( \frac{1}{4} - \frac{4}{3} + \frac{3}{2} \right) - (0) \\ &= \frac{5}{12} \end{aligned}$$

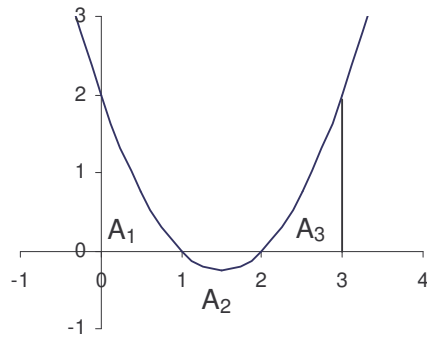
$$\text{Area } A_1 = \frac{5}{12} \text{ square units}$$

$$\begin{aligned} \int_1^3 (x^3 - 4x^2 + 3x) dx &= \left[ \frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2} \right]_1^3 \\ &= \left( \frac{3^4}{4} - \frac{4(3)^3}{3} + \frac{3(3)^2}{2} \right) - \left( \frac{1}{4} - \frac{4}{3} + \frac{3}{2} \right) \\ &= \left( \frac{81}{4} - 36 + \frac{27}{2} \right) - \left( \frac{5}{12} \right) \\ &= -\frac{9}{4} - \frac{5}{12} \\ &= -\frac{8}{3} \end{aligned}$$

$$\text{Area } A_2 = \frac{8}{3} \text{ square units}$$

$$\text{Total enclosed area} = A_1 + A_2 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} = 3\frac{1}{12} \text{ square units}$$

(b) Curve cuts the x-axis where:  $(x - 1)(x - 2) = 0 \Rightarrow x = 1, 2$



$$y = (x - 1)(x - 2) = x^2 - 3x + 2$$

$$\begin{aligned} \int_0^1 (x^2 - 3x + 2) dx &= \left[ \frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_0^1 \\ &= \left( \frac{1}{3} - \frac{3}{2} + 2 \right) - (0) \\ &= \frac{5}{6} \end{aligned}$$

$$\text{Area } A_1 = \frac{5}{6} \text{ square units}$$

$$\begin{aligned} \int_1^2 (x^2 - 3x + 2) dx &= \left[ \frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_1^2 \\ &= \left( \frac{8}{3} - \frac{12}{2} + 4 \right) - \left( \frac{1}{3} - \frac{3}{2} + 2 \right) \\ &= \frac{2}{3} - \frac{5}{6} \\ &= -\frac{1}{6} \end{aligned}$$

$$\text{Area } A_2 = \frac{1}{6} \text{ square units}$$

$$\begin{aligned} \int_2^3 (x^2 - 3x + 2) dx &= \left[ \frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_2^3 \\ &= \left( \frac{27}{3} - \frac{27}{2} + 6 \right) - \left( \frac{8}{3} - \frac{12}{2} + 4 \right) \\ &= \frac{3}{2} - \frac{2}{3} \\ &= \frac{5}{6} \end{aligned}$$

$$\text{Area } A_3 = \frac{5}{6} \text{ square units}$$

$$\text{Total enclosed area} = A_1 + A_2 + A_3 = \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6} = 1\frac{5}{6} \text{ square units}$$